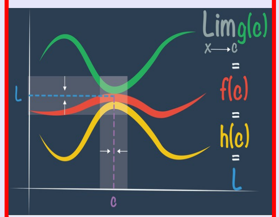
Calculus I

Lecture 6



Feb 19-8:47 AM

$$f(x) = x^2 - 2x + 3$$

Evaluate $\displaystyle\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$

$$\lim_{h\to 0} \frac{(a+h)^2 - 2(a+h) + 3 - a^2 + 2a - 3}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{a^2} + 2ah + h^2 \cancel{- 2a} - 2h \cancel{+3} \cancel{- a^2} \cancel{+ 2a} \cancel{- 3}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(2a + h - 2)}{\cancel{h}} = \lim_{h\to 0} (2a + \overset{\to 0}{\cancel{h}} - 2) = \boxed{2a - 2}$$

Jan 13-8:05 AM

Given $\lim\limits_{x \to a} [f(x) + g(x)] = 2$

$\lim\limits_{x \to a} [f(x) - g(x)] = 1$

$\Rightarrow \begin{cases} \lim\limits_{x \to a} f(x) + \lim\limits_{x \to a} g(x) = 2 \\ \lim\limits_{x \to a} f(x) - \lim\limits_{x \to a} g(x) = 1 \end{cases}$

Find $\lim\limits_{x \to a} [f(x) \cdot g(x)]$. $= \lim\limits_{x \to a} f(x) \cdot \lim\limits_{x \to a} g(x)$

$= \frac{3}{2} \cdot \frac{1}{2} = \boxed{\frac{3}{4}}$

By addition Rule

$2 \lim\limits_{x \to a} f(x) = 3$

$\lim\limits_{x \to a} f(x) = \frac{3}{2}$

$\frac{3}{2} + \lim\limits_{x \to a} g(x) = 2$

$\lim\limits_{x \to a} g(x) = \frac{1}{2}$

Jan 13-8:09 AM

Evaluate $\lim\limits_{x \to 1} \frac{\sqrt[3]{x} - 1}{\sqrt{x} - 1} = \frac{\sqrt[3]{1} - 1}{\sqrt{1} - 1} = \frac{1-1}{1-1} = \frac{0}{0}$

I.F.

index 3

index 2 $\to 6$

Let $x = t^{\boxed{6}}$

as $x \to 1$, $t \to 1$

$\lim\limits_{x \to 1} \frac{\sqrt[3]{x} - 1}{\sqrt{x} - 1} = \lim\limits_{t \to 1} \frac{\sqrt[3]{t^6} - 1}{\sqrt{t^6} - 1} = \lim\limits_{t \to 1} \frac{t^2 - 1}{t^3 - 1}$

$= \lim\limits_{t \to 1} \frac{\cancel{(t-1)}(t+1)}{\cancel{(t-1)}(t^2 + t + 1)} = \lim\limits_{t \to 1} \frac{t+1}{t^2 + t + 1} = \boxed{\frac{2}{3}}$

Jan 13-8:15 AM

Find $f'(x)$ for $f(x) = x^4$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^4 - x^4}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{1x^4} + 4x^3h + 6x^2h^2 + 4xh^3 + 1h^4 \cancel{- x^4}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(4x^3 + \cancel{6x^2h + 4xh^2 + h^3})}{\cancel{h}} \to 0 \quad = \boxed{4x^3}$$

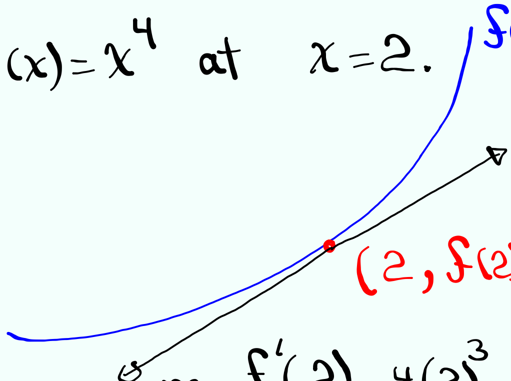
Jan 13-8:22 AM

Find eqn of the tan. line to the graph of $f(x) = x^4$ at $x = 2$.

$f(x) = x^4$

$f(2) = 2^4 = 16$

$\to f'(x) = 4x^3$



$(2, f(2)) = (2, 16)$

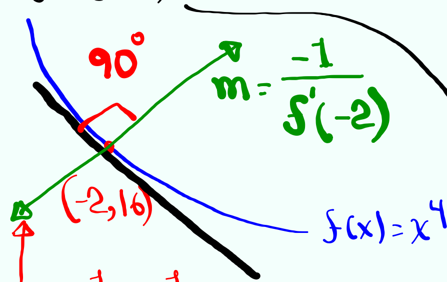
$m = f'(2) = 4(2)^3 = 4(8) = \boxed{32}$

$y - y_1 = m(x - x_1)$

$y - 16 = 32(x - 2) \to \boxed{y = 32x - 48}$

Jan 13-8:28 AM

Find equation of the normal line to the graph of $f(x) = x^4$ at $x = -2$.



$m = \dfrac{-1}{f'(-2)}$

$f(-2) = (-2)^4 = 16$

$f'(x) = 4x^3$

$f'(-2) = 4(-2)^3$
$= -32$

$m = \dfrac{-1}{-32} = \dfrac{1}{32}$

$y - y_1 = m(x - x_1)$

$y - 16 = \frac{1}{32}(x - -2)$

$y = \frac{1}{32}x + \frac{2}{32} + 16$

$y = \frac{1}{32}x + \frac{1}{16} + 16$

$\boxed{y = \frac{1}{32}x + \frac{257}{16}}$

Jan 13-8:32 AM

Find all values of a that makes the function below continuous on all real numbers.

$\lim_{x \to a} f(x) = f(a)$

$= a^2$

$f(x) = \begin{cases} \underbrace{x+2}_{\text{Cont. everywhere}} & \text{if } x < a \\ \underbrace{x^2}_{\text{Cont. everywhere}} & \text{if } x \geq a \end{cases}$

$\lim_{x \to a^-} f(x) = a + 2$

$\lim_{x \to a^+} f(x) = a^2$

$\Rightarrow a^2 = a + 2$

$a^2 - a - 2 = 0$

$(a-2)(a+1) = 0$

$\boxed{a = 2}$ $\boxed{a = -1}$

Jan 13-8:38 AM

Find c such that $\lim\limits_{x\to0}\frac{\sqrt[3]{1+cx}-1}{x}=\boxed{1.}$

$$=\frac{\sqrt[3]{1+c(0)}-1}{0}=\frac{1-1}{0}=\frac{0}{0}$$

I.F.

Let $t=\sqrt[3]{1+cx}$ $x\to0$, $t\to1$

$t^3=1+cx \longrightarrow x=\frac{t^3-1}{c}$ if $c\neq0$

$$\lim_{x\to0}\frac{\sqrt[3]{1+cx}-1}{x}=\lim_{t\to1}\frac{t-1}{\frac{t^3-1}{c}}$$

$$=\lim_{t\to1}\frac{c(t-1)}{t^3-1}=\lim_{t\to1}\frac{c\cancel{(t-1)}}{\cancel{(t-1)}(t^2+t+1)}$$

$$=\frac{c}{1^2+1+1}=\frac{c}{3}$$

$\frac{c}{3}=1 \rightarrow \boxed{c=3}$

Jan 13-8:45 AM

Two Facts:

1) $\lim\limits_{h\to0}\frac{\sin h}{h}=1$

2) $\lim\limits_{h\to0}\frac{\cos h-1}{h}=0$

We will Prove them.

Evaluate $\lim\limits_{x\to0}\frac{5\sin 5x}{5x}=5\lim\limits_{x\to0}\frac{\sin 5x}{5x}\to1$

$=5\cdot1=\boxed{5}$

Let $x=.001$

Evaluate $\frac{\sin 5(.001)}{.001}\approx4.999979167$

what about $x=.0001$

$$\frac{\sin 5(.0001)}{.0001}\approx4.999999792$$

as $x\to0$, $\frac{\sin 5x}{x}\to5$

Jan 13-8:53 AM

Evaluate $\lim_{x \to 0} \frac{\sin 3x}{\sin 4x} = \frac{\sin 0}{\sin 0} = \frac{0}{0}$ I.F.

as $x \to 0$ $\frac{\sin 3x}{\sin 4x} \to .75$

Let $x = .0001$

$\frac{\sin 3(.0001)}{\sin 4(.0001)} \approx .75000000 88$

try $x = .00001$

$\frac{\sin 3(.00001)}{\sin 4(.00001)} \approx .7500000001$

$$\lim_{x \to 0} \frac{\sin 3x}{\sin 4x} = \lim_{x \to 0} \frac{\frac{3 \sin 3x}{3x}}{\frac{4 \sin 4x}{4x}} = \frac{3}{4} \cdot \frac{\cancel{\lim_{x \to 0} \frac{\sin 3x}{3x}} \to 1}{\cancel{\lim_{x \to 0} \frac{\sin 4x}{4x}} \to 1}$$

$$= \frac{3}{4} \cdot \frac{1}{1} = \boxed{\frac{3}{4}}$$

.75

Jan 13-9:00 AM

Evaluate $\lim_{x \to 1} \frac{\sin(x-1)}{x^2 - 1} = \frac{\sin(1-1)}{1^2 - 1} = \frac{\sin 0}{0} = \frac{0}{0}$

I.F.

I want to use $\lim_{h \to 0} \frac{\sin h}{h} = 1$

$$= \lim_{x \to 1} \frac{\sin(x-1)}{(x+1)(x-1)}$$

$$= \lim_{x \to 1} \frac{1}{x+1} \cdot \cancel{\lim_{x \to 1} \frac{\sin(x-1)}{x-1}} \to 1$$

$$= \frac{1}{1+1} \cdot 1 = \boxed{\frac{1}{2}}$$

$x = 1.001$

$\frac{\sin(1.001 - 1)}{1.001^2 - 1} \approx .4997...$

$\approx .5$

Jan 13-9:08 AM

Evaluate $\lim_{x \to 0} \frac{\cos x - 1}{\sin x} = \frac{\cos 0 - 1}{\sin 0} = \frac{1-1}{0} = \frac{0}{0}$

I.F.

Let $x = .001$

$\frac{\cos .001 - 1}{\sin .001} = -5 \times 10^{-4} = -.0005$

as $x \to 0$ $\frac{\cos x - 1}{\sin x} \to 0$

We know

$\lim_{x \to 0} \frac{\sin x}{x} = 1$

$\lim_{x \to 0} \frac{\cos x - 1}{x} = 0$

$$\lim_{x \to 0} \frac{\cos x - 1}{\sin x}$$

$$= \lim_{x \to 0} \frac{\frac{\cos x - 1}{x}}{\frac{\sin x}{x}}$$

$$= \frac{\lim_{x \to 0} \frac{\cos x - 1}{x} \to 0}{\lim_{x \to 0} \frac{\sin x}{x} \to 1} = \frac{0}{1} = \boxed{0}$$

Jan 13-9:17 AM

Evaluate $\lim_{x \to 0} \frac{\sin 3x \cdot \sin 5x}{x^2} = \frac{\sin 0 \cdot \sin 0}{0^2} = \frac{0}{0}$

I.F.

No rule

$\sin 3x = 3 \sin x$

$$= \lim_{x \to 0} \frac{3 \sin 3x}{3x} \cdot \lim_{x \to 0} \frac{5 \sin 5x}{5x}$$

$$= 3 \lim_{x \to 0} \frac{\sin 3x}{3x} \to 1 \cdot 5 \lim_{x \to 0} \frac{\sin 5x}{5x} \to 1$$

$$= 3 \cdot 1 \cdot 5 \cdot 1 = \boxed{15}$$

Jan 13-9:24 AM

More on derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Other notation

$y = f(x)$

$y' = f'(x)$

$y' = \frac{d}{dx}[f(x)]$

$y' = \frac{dy}{dx}$

Jan 13-9:56 AM

Some rules:

1) $\frac{d}{dx}[c] = 0$

$f(x) = 5$ $\quad f'(x) = 0$

2) $\frac{d}{dx}[x] = 1$

$f(x) = x$ $\quad f'(x) = 1$

3) $\frac{d}{dx}[x^n] = n x^{n-1}$

$f(x) = x^5$ $\quad f'(x) = 5x^4$

4) $\frac{d}{dx}[c f(x)] = c \frac{d}{dx}[f(x)]$

$f(x) = -3x^4$

$f'(x) = -3 \cdot 4x^3$

$= \boxed{-12x^3}$

5) $\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$

$$\frac{d}{dx}[x^3 + x + 5] = \frac{d}{dx}[x^3] + \frac{d}{dx}[x] + \frac{d}{dx}[5]$$

$$= 3x^2 + 1 + 0 = \boxed{3x^2 + 1}$$

Jan 13-10:00 AM

6) $\frac{d}{dx}\left[f(x) - g(x)\right] = \frac{d}{dx}\left[f(x)\right] - \frac{d}{dx}\left[g(x)\right]$

$f(x) = 5x^3 - 8x^2 - 12$

$f'(x) = \frac{d}{dx}\left[5x^3 - 8x^2 - 12\right]$

$= \frac{d}{dx}\left[5x^3\right] - \frac{d}{dx}\left[8x^2\right] - \cancel{\frac{d}{dx}\left[12\right]} \rightarrow 0$

$= 5\frac{d}{dx}x^3 - 8\frac{d}{dx}x^2 - 0$

$= 5 \cdot 3x^2 - 8 \cdot 2x = \boxed{15x^2 - 16x}$

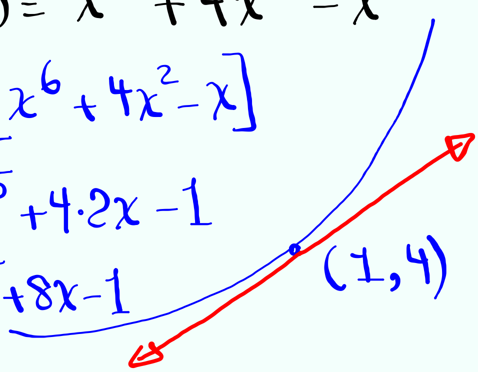
Jan 13-10:07 AM

find slope of the tan. line to the graph of $f(x) = x^6 + 4x^2 - x$ at $x = 1$.

$f'(x) = \frac{d}{dx}\left[x^6 + 4x^2 - x\right]$

$= 6x^5 + 4 \cdot 2x - 1$

$= 6x^5 + 8x - 1$



$m = f'(1)$

$= 6(1)^5 + 8(1) - 1$

$= \boxed{13}$

Jan 13-10:11 AM

find slope of the normal line to the graph of $f(x) = x - \frac{1}{x}$ at $x=1$.

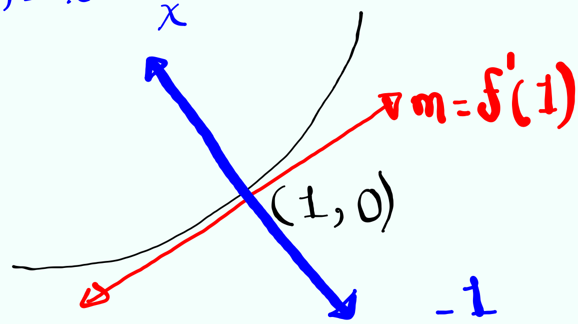
$f(x) = x - \frac{1}{x}$

$f(x) = x - \boxed{x^{-1}}$

$f'(x) = 1 - (-1)x^{-1-1}$

$= 1 + x^{-2} = 1 + \frac{1}{x^2}$

$f'(1) = 1 + \frac{1}{1^2} = 1 + 1 = 2$



$m = \frac{-1}{f'(1)}$

$\boxed{m = \frac{-1}{2}}$

Jan 13-10:15 AM

More rules

7) $\frac{d}{dx}\left[f(x) \cdot g(x)\right] = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

Product Rule

$\frac{d}{dx}\left[(x^2+4x)(x^3-2)\right] = (2x+4) \cdot (x^3-2) + (x^2+4x) \cdot 3x^2$

$= 2x^4 - 4x + 4x^3 - 8 + 3x^4 + 12x^3$

$= \boxed{5x^4 + 16x^3 - 4x - 8}$

Suppose $f(3) = 5$ $\quad f'(3) = 2$

$g(3) = 4$ $\quad g'(3) = -1$

Find $\boxed{\frac{d}{dx}\left[f(x) \cdot g(x)\right]}$, then evaluate at $x=3$.

$f'(x) \cdot g(x) + f(x) \cdot g'(x)$

$= f'(3) \cdot g(3) + f(3) \cdot g'(3) = 2 \cdot 4 + 5 \cdot (-1)$

$= 8 - 5 = \boxed{3}$

Jan 13-10:21 AM

$f(2) = 6$, $f'(2) = -4$

Find $\boxed{\frac{d}{dx}\left[x^3 f(x)\right]}$, then evaluate at $x=2$.

$= 3x^2 \cdot f(x) + x^3 \cdot f'(x)$

$= 3(2)^2 f(2) + 2^3 \cdot f'(2)$

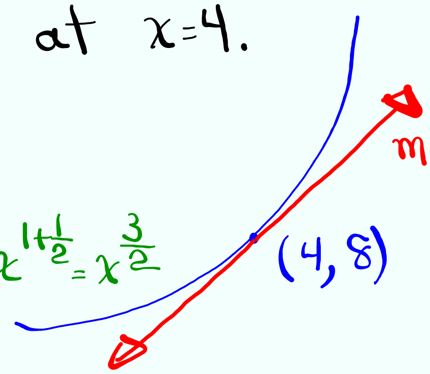
$= 3 \cdot 4 \cdot 6 + 8 \cdot (-4)$

$= 72 - 32 = \boxed{40}$

Jan 13-10:31 AM

Find eqn of the tan. line to the graph of $f(x) = x\sqrt{x}$ at $x=4$.

$f(4) = 4\sqrt{4} = \boxed{8}$



$f(x) = x\sqrt{x}$

Algebra $\to x^1 \cdot x^{1/2} = x^{1+\frac{1}{2}} = x^{\frac{3}{2}}$

$f(x) = x^{3/2}$

$f'(x) = \frac{3}{2}x^{\frac{3}{2}-1}$

$= \frac{3}{2}x^{1/2} = \frac{3}{2}\sqrt{x}$

$m = f'(4)$

$= \frac{3}{2}\sqrt{4}$

$= \frac{3}{2} \cdot 2$

$= \boxed{3}$

$y - 8 = 3(x-4)$

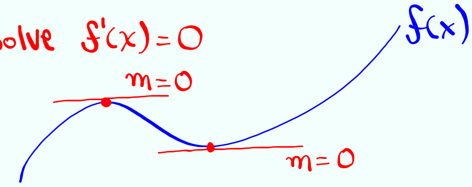
$\boxed{y = 3x - 4}$

Jan 13-10:35 AM

Find x-value on the graph of $f(x)=x^3+3x^2+x+3$

where tan. line is horizontal.

Solve $f'(x)=0$



$f'(x)=3x^2+6x+1$

Solve $3x^2+6x+1=0$

$a=3 \quad b=6 \quad c=1$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-6\pm\sqrt{36-4(3)(1)}}{2(3)}$$

$$=\frac{-6\pm\sqrt{24}}{6}$$

$x=\frac{-6+\sqrt{24}}{6}$

$x=\frac{-6-\sqrt{24}}{6}$

Jan 13-10:46 AM

More rule

8) $$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{f'(x)\cdot g(x)-f(x)\cdot g'(x)}{[g(x)]^2}$$

Quotient Rule

$$\frac{d}{dx}\left[\frac{2x-1}{3x+4}\right]=\frac{2(3x+4)-(2x-1)\cdot 3}{(3x+4)^2}$$

$$=\frac{6x+8-6x+3}{(3x+4)^2}=\frac{11}{(3x+4)^2}$$

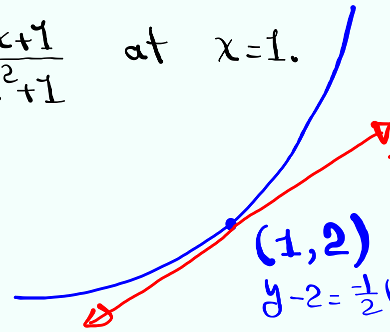
$$\frac{d}{dx}\left[\frac{x^3}{x^2-1}\right]=\frac{3x^2(x^2-1)-x^3(2x)}{(x^2-1)^2}$$

$$=\frac{3x^4-3x^2-2x^4}{(x^2-1)^2}=\boxed{\frac{x^4-3x^2}{(x^2-1)^2}}$$

Jan 13-10:53 AM

Find eqn of the tan. line to the graph of $f(x)=\dfrac{3x+1}{x^2+1}$ at $x=1$.

$f(1)=2$ ✓



$m=f'(1)$

$=\boxed{\dfrac{-1}{2}}$

$(1,2)$

$y-2=\frac{-1}{2}(x-1)$

$\boxed{y=\frac{-1}{2}x+\frac{5}{2}}$

$$f(x)=\frac{3x+1}{x^2+1}$$

$$f'(x)=\frac{3(x^2+1)-(3x+1)\cdot 2x}{(x^2+1)^2}=\frac{3x^2+3-6x^2-2x}{(x^2+1)^2}$$

$$=\frac{-3x^2-2x+3}{(x^2+1)^2}$$

$$f'(1)=\frac{-3\cdot 1^2-2\cdot 1+3}{(1^2+1)^2}=\frac{-2}{4}=\boxed{\frac{-1}{2}}$$

Jan 13-11:02 AM

$f(5)=1 \qquad f'(5)=6$

$g(5)=-3 \qquad g'(5)=2$

Find $h'(5)$ where $h(x)=\dfrac{f(x)}{g(x)}$

$$h'(x)=\frac{f'(x)\cdot g(x)-f(x)\cdot g'(x)}{[g(x)]^2}$$

$$h'(5)=\frac{f'(5)\cdot g(5)-f(5)g'(5)}{[g(5)]^2}$$

$$=\frac{6\cdot(-3)-1\cdot 2}{(-3)^2}$$

$$=\frac{-18-2}{9}=\boxed{\frac{-20}{9}}$$

Jan 13-11:10 AM

Evaluate $\lim_{x \to 1} \frac{x^{100} - 1}{x - 1} = \frac{1^{100} - 1}{1 - 1} = \frac{0}{0}$ I.F.

Hint:
$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$a = 1$

$f(x) = x^{100}$

$f(1) = 1$

$f'(x) = 100x^{99}$

$= 100(1)^{99} = \boxed{100}$

Jan 13-11:16 AM

Evaluate $\lim_{x \to \frac{\pi}{4}} \frac{\tan x - 1}{x - \frac{\pi}{4}}$

Identify $f(x)$ and a using
$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$a = \frac{\pi}{4}$

$f(x) = \tan x$

$f(\frac{\pi}{4}) = \tan \frac{\pi}{4} = 1$

You will see $\frac{d}{dx}[\tan x] = \sec^2 x$

$= \sec^2 \frac{\pi}{4}$

$= \frac{1}{\cos^2 \frac{\pi}{4}}$

$= \frac{1}{\left(\frac{\sqrt{2}}{2}\right)^2} = \frac{1}{\frac{2}{4}} = \frac{1}{\frac{1}{2}} = \boxed{2}$

Jan 13-11:23 AM